A study on the variety of minimal rational tangents and its application

Hosung Kim

Korea Institute for Advanced Study

2012.12.20.

Contents

Let X be a Fano complex manifold(i.e., $-K_X$ is ample) with Picard number 1. Let $n := \dim X$.

Definition

A **rational curve** C on X is the image of a morphism $f : \mathbb{P}^1 \to X$ which is birational over its image.
The morphism f is called a parametrization of C.

Theorem (S.Mori, 1979)

X is uniruld, that is, for any point $x \in X$, there exists a rational curve passing through x.

Let X be a Fano complex manifold(i.e., $-K_X$ is ample) with Picard number 1. Let $n := \dim X$.

Definition

A **rational curve** C on X is the image of a morphism $f : \mathbb{P}^1 \to X$ which is birational over its image.
The morphism f is called a parametrization of C.

Theorem (S.Mori, 1979)

X is unirulD, that is, for any point $x \in X$, there exists a rational curve passing through x.

Let X be a Fano complex manifold(i.e., $-K_X$ is ample) with Picard number 1. Let $n := \dim X$.

Definition

A **rational curve** C on X is the image of a morphism $f : \mathbb{P}^1 \to X$ which is birational over its image.
The morphism f is called a parametrization of C.

Theorem (S.Mori, 1979)

X is unirulded, that is, for any point $x \in X$, there exists a rational curve passing through x.

Let $C \subset X$ be a rational curve parametrized by $f : \mathbb{P}^1 \to X$.
Set

$$f^*TX = \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i).$$

Assume that $a_i \geq -1$ for all i.
Then the union of the locus of the curves on X which can be obtained by a deformation of C has dimension $\#\{a_i \geq 0\}$.

$$\dim \bigcup_{C \rightsquigarrow C'} C' = \#\{a_i \geq 0\}$$

Let $C \subset X$ be a rational curve parametrized by $f : \mathbb{P}^1 \to X$. Set

$$f^*TX = \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i).$$

Assume that $a_i \geq -1$ for all i.

Then the union of the locus of the curves on X which can be obtained by a deformation of C has dimension $\#\{a_i \geq 0\}$.

$$\dim \bigcup_{C \rightsquigarrow C'} C' = \#\{a_i \geq 0\}$$

Let $C \subset X$ be a rational curve parametrized by $f : \mathbb{P}^1 \to X$. Set

$$f^*TX = \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i).$$

Assume that $a_i \geq -1$ for all i.
Then the union of the locus of the curves on X which can be obtained by a deformation of C has dimension $\#\{a_i \geq 0\}$.

$$\dim \bigcup_{C \rightsquigarrow C'} C' = \#\{a_i \geq 0\}$$

Definition

A **free rational curve** on X is a rational curve parametrized by $f : \mathbb{P}^1 \to X$ with

$$f^* T_X = \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i), \ a_i \geq 0.$$

Theorem

For general $x \in X$, any rational curve through x is free.

Definition

A **minimal free rational curve** on X is a free rational curve C with minimal anti-canonical degree $(-K_X) \cdot C$.

Definition

A **free rational curve** on X is a rational curve parametrized by $f : \mathbb{P}^1 \to X$ with

$$f^*T_X = \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i),\ \ a_i \geq 0.$$

Theorem

For general $x \in X$, any rational curve through x is free.

Definition

A **minimal free rational curve** on X is a free rational curve C with minimal anti-canonical degree $(-K_X) \cdot C$.

Definition

A **free rational curve** on X is a rational curve parametrized by $f : \mathbb{P}^1 \to X$ with

$$f^* T_X = \oplus_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(a_i), \;\; a_i \geq 0.$$

Theorem

For general $x \in X$, any rational curve through x is free.

Definition

A **minimal free rational curve** on X is a free rational curve C with minimal anti-canonical degree $(-K_X) \cdot C$.

Definition

Let x be a general point of X.
The **variety of minimal rational curves at** x is the normalization of the space of all minimal free rational curves on X through x, and we denote it by $\mathcal{K}_x$.

$$\mathcal{K}_x := \{\text{minimal free rational curves through } x\}^n$$

- Fix a point $0 \in \mathbb{P}^1$.
  $\mathcal{K}_x$ is isomorphic to the union of several irreducible components of

  $$\mathrm{Hom}_{bir}(\mathbb{P}^1, X, 0 \mapsto x)/\mathrm{Aut}(\mathbb{P}^1, 0).$$

  So every point in $\mathcal{K}_x$ can be represented by a birational morphism $f : \mathbb{P}^1 \to X$ with $f(0) = x$.
- $\mathcal{K}_x$ is a smooth projective variety of dimension p where $(-K_X) \cdot C = p + 2$ for $[C] \in \mathcal{K}$.

- Fix a point $0 \in \mathbb{P}^1$.
  $\mathcal{K}_x$ is isomorphic to the union of several irreducible components of

  $$\mathrm{Hom}_{bir}(\mathbb{P}^1, X, 0 \mapsto x)/\mathrm{Aut}(\mathbb{P}^1, 0).$$

  So every point in $\mathcal{K}_x$ can be represented by a birational morphism $f : \mathbb{P}^1 \to X$ with $f(0) = x$.
- $\mathcal{K}_x$ is a smooth projective variety of dimension p where $(-K_X) \cdot C = p + 2$ for $[C] \in \mathcal{K}$.

- Fix a point $0 \in \mathbb{P}^1$.
  $\mathcal{K}_x$ is isomorphic to the union of several irreducible components of

$$\mathrm{Hom}_{bir}(\mathbb{P}^1, X, 0 \mapsto x)/\mathrm{Aut}(\mathbb{P}^1, 0).$$

  So every point in $\mathcal{K}_x$ can be represented by a birational morphism $f : \mathbb{P}^1 \to X$ with $f(0) = x$.
- $\mathcal{K}_x$ is a smooth projective variety of dimension p where $(-K_X) \cdot C = p + 2$ for $[C] \in \mathcal{K}$.

Define the rational map

$$\tau_x : \mathcal{K}_x \dashrightarrow \mathbb{P}(T_xX), \quad \text{by } [C] \mapsto \mathbb{P}(T_xC)$$

which is called the tangent map.

Definition

The **variety of minimal rational tangents (VMRT)** at x is

$$\mathcal{C}_x := \overline{\mathrm{Im}\tau_x}.$$

Define the rational map

$$\tau_x : \mathcal{K}_x \dashrightarrow \mathbb{P}(T_x X), \quad \text{by } [C] \mapsto \mathbb{P}(T_x C)$$

which is called the tangent map.

Definition

The **variety of minimal rational tangents (VMRT)** at x is

$$\mathcal{C}_x := \overline{\mathrm{Im}\tau_x}.$$

Theorem (S. Kebekus, 2002)

τ_x can be extended to a finite morphism.

In fact, any morphism $[f : \mathbb{P}^1 \to X, 0 \mapsto x] \in \mathcal{K}_x$ is an immersion at 0, and thus define $\tau([f]) := \mathbb{P}df(T_0\mathbb{P}^1)$.

Theorem (J.-M. Hwang and N.Mok, 2004)

The tangent morphism $\tau_x : \_mathcal K_x \to \mathcal{C}_x$ is birational, and thus it is the normalization morphism of $\mathcal{C}_x$.

Conjecture

The tangent morphism $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ is an isomorphism.

Theorem (S. Kebekus, 2002)

τ_x can be extended to a finite morphism.

In fact, any morphism $[f : \mathbb{P}^1 \to X, 0 \mapsto x] \in \mathcal{K}_x$ is an immersion at 0, and thus define $\tau([f]) := \mathbb{P}df(T_0\mathbb{P}^1)$.

Theorem (J.-M. Hwang and N.Mok, 2004)

The tangent morphism $\tau_x : \_mathcal K_x \to \mathcal{C}_x$ is birational, and thus it is the normalization morphism of $\mathcal{C}_x$.

Conjecture

The tangent morphism $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ is an isomorphism.

Theorem (S. Kebekus, 2002)

τ_x can be extended to a finite morphism.

In fact, any morphism $[f : \mathbb{P}^1 \to X, 0 \mapsto x] \in \mathcal{K}_x$ is an immersion at 0, and thus define $\tau([f]) := \mathbb{P}df(T_0\mathbb{P}^1)$.

Theorem (J.-M. Hwang and N.Mok, 2004)

The tangent morphism $\tau_x : \_mathcal K_x \to \mathcal{C}_x$ is birational, and thus it is the normalization morphism of $\mathcal{C}_x$.

Conjecture

The tangent morphism $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ is an isomorphism.

Theorem (S. Kebekus, 2002)

τ_x can be extended to a finite morphism.

In fact, any morphism $[f : \mathbb{P}^1 \to X, 0 \mapsto x] \in \mathcal{K}_x$ is an immersion at 0, and thus define $\tau([f]) := \mathbb{P}df(T_0\mathbb{P}^1)$.

Theorem (J.-M. Hwang and N.Mok, 2004)

The tangent morphism $\tau_x : \_mathcal K_x \to \mathcal{C}_x$ is birational, and thus it is the normalization morphism of $\mathcal{C}_x$.

Conjecture

The tangent morphism $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ is an isomorphism.

Proposition

The tangent morphism τ_x *is an immersion at* $[f : \mathbb{P}^1 \to X, 0 \mapsto x] \in \mathcal{K}_x$ *if and only if*

$$f^*T_X = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^p \oplus \mathcal{O}_{\mathbb{P}^1}^{n-1-p}.$$

Theorem (J.-M. Hwang, 2001)

Suppose that $X \subset \mathbb{P}^N$ and for each point $x \in X$, there exists a line through x in $\mathbb{P}^N$ lying on X. Then for general $x \in X$, the tangent morphism $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ is an isomorphism.

The projective geometry of $\mathcal{C}_x$ gives a hint on the geometry of X.

Question

- What are defining equations of $\mathcal{C}_x \subset \mathbb{P}(T_xX)$?
- How varies the projective isomorphism type of $\mathcal{C}_x$?

Let $X = \mathbb{P}^n$.

Then minimal free rational curves are lines.

For a line $\ell \subset \mathbb{P}^n$,

$$T_{\mathbb{P}^n}|_\ell = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(n-1)}.$$

Since $-K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(n+1)$, the anti-canonical degree

$$-K_{\mathbb{P}^n} \cdot \ell = n + 1$$

is minimal. Moreover,

$$\mathcal{K}_x \cong \mathcal{C}_x = \mathbb{P}T_x(\mathbb{P}^n) \cong \mathbb{P}^{n-1}.$$

Let $X = \mathbb{P}^n$.

Then minimal free rational curves are lines.

For a line $\ell \subset \mathbb{P}^n$,

$$T_{\mathbb{P}^n}|_\ell = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(n-1)}.$$

Since $-K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(n+1)$, the anti-canonical degree

$$-K_{\mathbb{P}^n} \cdot \ell = n + 1$$

is minimal. Moreover,

$$\mathcal{K}_x \cong \mathcal{C}_x = \mathbb{P}T_x(\mathbb{P}^n) \cong \mathbb{P}^{n-1}.$$

Let $X = \mathbb{P}^n$.
Then minimal free rational curves are lines.
For a line $\ell \subset \mathbb{P}^n$,

$$T_{\mathbb{P}^n}|_\ell = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(n-1)}.$$

Since $-K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(n+1)$, the anti-canonical degree

$$-K_{\mathbb{P}^n} \cdot \ell = n+1$$

is minimal. Moreover,

$$\mathcal{K}_x \cong \mathcal{C}_x = \mathbb{P}T_x(\mathbb{P}^n) \cong \mathbb{P}^{n-1}.$$

Let $X = \mathbb{P}^n$.
Then minimal free rational curves are lines.
For a line $\ell \subset \mathbb{P}^n$,

$$T_{\mathbb{P}^n}|_\ell = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(n-1)}.$$

Since $-K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(n+1)$, the anti-canonical degree

$$-K_{\mathbb{P}^n} \cdot \ell = n + 1$$

is minimal. Moreover,

$$\mathcal{K}_x \cong \mathcal{C}_x = \mathbb{P}T_x(\mathbb{P}^n) \cong \mathbb{P}^{n-1}.$$

Let $X = \mathbb{P}^n$.
Then minimal free rational curves are lines.
For a line $\ell \subset \mathbb{P}^n$,

$$T_{\mathbb{P}^n}|_\ell = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus(n-1)}.$$

Since $-K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(n+1)$, the anti-canonical degree

$$-K_{\mathbb{P}^n} \cdot \ell = n + 1$$

is minimal. Moreover,

$$\mathcal{K}_x \cong \mathcal{C}_x = \mathbb{P}T_x(\mathbb{P}^n) \cong \mathbb{P}^{n-1}.$$

Let $X = X_m \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree m, $2 \leq m \leq n$.

The minimal free rational curves are lines on X, and

$$\mathcal{K}_x \cong \mathcal{C}_x \subset \mathbb{P}(T_x X)$$

is a smooth complete intersection of degree $(2, ..., m)$.

Assume $x = (1 : 0 : \cdots : 0) \in X$, and let $f(t_0, ..., t_{n+1}) = 0$ be the defining equation of X.
The line through x and $y = (0 : y_1 : \cdots : y_{n+1})$ lies on X.

$$\Leftrightarrow f(1, \lambda y_1 : \cdots \lambda y_{n+1}) = b_1(y)\lambda + b_2(y)\lambda^2 + \cdots + b_m(y)\lambda^m = 0$$
$$\Leftrightarrow b_1(y) = b_2(y) = \cdots = b_m(y) = 0.$$

Let $X = X_m \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree m, $2 \leq m \leq n$.
The minimal free rational curves are lines on X, and

$$\mathcal{K}_x \cong \mathcal{C}_x \subset \mathbb{P}(T_x X)$$

is a smooth complete intersection of degree $(2, ..., m)$.

Assume $x = (1 : 0 : \cdots : 0) \in X$, and let $f(t_0, ..., t_{n+1}) = 0$ be the defining equation of X.
The line through x and $y = (0 : y_1 : \cdots : y_{n+1})$ lies on X.

$$\Leftrightarrow f(1, \lambda y_1 : \cdots \lambda y_{n+1}) = b_1(y)\lambda + b_2(y)\lambda^2 + \cdots + b_m(y)\lambda^m = 0$$

$$\Leftrightarrow b_1(y) = b_2(y) = \cdots = b_m(y) = 0.$$

Let $X = X_m \subset \mathbb{P}^{n+1}$ be a smooth hypersurface of degree m, $2 \leq m \leq n$.
The minimal free rational curves are lines on X, and

$$\mathcal{K}_x \cong \mathcal{C}_x \subset \mathbb{P}(T_x X)$$

is a smooth complete intersection of degree $(2, ..., m)$.

Assume $x = (1 : 0 : \cdots : 0) \in X$, and let $f(t_0, ..., t_{n+1}) = 0$ be the defining equation of X.
The line through x and $y = (0 : y_1 : \cdots : y_{n+1})$ lies on X.

$$\Leftrightarrow f(1, \lambda y_1 : \cdots \lambda y_{n+1}) = b_1(y)\lambda + b_2(y)\lambda^2 + \cdots + b_m(y)\lambda^m = 0$$
$$\Leftrightarrow b_1(y) = b_2(y) = \cdots = b_m(y) = 0.$$

Theorem (Lansberg and Robles, 2010)

Let $X \subset \mathbb{P}^{n+1}$ be a general smooth hypersurface of degree m, $2 \leq m \leq n$. Then the projective isomorphism type of $\mathcal{C}_x$ varies in a maximal way as x moves over general points of X

Let $X = X_{d_1,\dots,d_c} \subset \mathbb{P}^N$ be a smooth complete intersection of degree $(d_1, \dots, d_c)$ with $d_i \geq 2$ and $\sum_{i=1}^{c}(d_i - 1) \leq n - 1$.

The minimal free rational curves are lines on X, and

$$\mathcal{K}_x \cong \mathcal{C}_x \subset \mathbb{P}(T_x X)$$

is a smooth complete intersection of degree

$$\begin{gathered} 2, 3, \dots, d_1, \\ 2, 3, \dots, d_2, \\ \vdots \\ 2, 3, \dots, d_c. \end{gathered}$$

Let $X = X_{d_1,\dots,d_c} \subset \mathbb{P}^N$ be a smooth complete intersection of degree $(d_1,\dots,d_c)$ with $d_i \geq 2$ and $\sum_{i=1}^{c}(d_i - 1) \leq n - 1$. The minimal free rational curves are lines on X, and

$$\mathcal{K}_x \cong \mathcal{C}_x \subset \mathbb{P}(T_x X)$$

is a smooth complete intersection of degree

$$\begin{aligned} &2, 3, \dots, d_1, \\ &2, 3, \dots, d_2, \\ &\vdots \\ &2, 3, \dots, d_c. \end{aligned}$$

Hartshorne Conjecture on complete intersection

Let $X \subset \mathbb{P}^N$ be an irreducible nondegenerate projective manifold of dimension n, and set $c := \mathrm{Codim}(X, \mathbb{P}^N)$. If $n \geq 2c + 1$, then X is a complete intersection.

Let $X \subset \mathbb{P}^N$ be defined by the intersection of m hypersurfaces of degrees $d_1 \geq \cdots \geq d_m$ where m is minimal.
Set $c := \mathrm{Codim}(X, \mathbb{P}^N)$ and $d := \sum_{i=1}^{c}(d_i - 1)$.

Theorem (Ionescu and Russo, 2011)

Assume $d \leq n-1$. Assume moreover that $n \geq c+2$ if X is a quadratic.
Then $X \subset \mathbb{P}^N$ is a complete intersection if and only if $\mathcal{C}_x \subset \mathbb{P}(T_x X)$ is a complete intersection of codimension d.
If $n \geq 2c+1$ and X is a quadratic, then X is a complete intersection.

Let $X \subset \mathbb{P}^N$ be defined by the intersection of m hypersurfaces of degrees $d_1 \geq \cdots \geq d_m$ where m is minimal.
Set $c := \mathrm{Codim}(X, \mathbb{P}^N)$ and $d := \sum_{i=1}^{c}(d_i - 1)$.

Theorem (Ionescu and Russo, 2011)

Assume $d \leq n-1$. Assume moreover that $n \geq c+2$ if X is a quadratic.
Then $X \subset \mathbb{P}^N$ is a complete intersection if and only if $\mathcal{C}_x \subset \mathbb{P}(T_x X)$ is a complete intersection of codimension d.
If $n \geq 2c+1$ and X is a quadratic, then X is a complete intersection.

Definition

Let $Y \subset \mathbb{P}^n$ be a smooth hypersurface of degree $2m$, $2 \leq m \leq n-1$. Let

$$\phi : X \to \mathbb{P}^n$$

be the double cover branched along Y.

Questions

- What are the minimal rational curves on X?
- Is τ_x an isomorphism?
- What are the defining equations of $\mathcal{C}_x \subset \mathbb{P}(T_x X)$?
- How varies $\mathcal{C}_x$?

Definition

Let $Y \subset \mathbb{P}^n$ be a smooth hypersurface of degree $2m$, $2 \leq m \leq n-1$. Let

$$\phi : X \to \mathbb{P}^n$$

be the double cover branched along Y.

Questions

- What are the minimal rational curves on X?
- Is τ_x an ismorphism?
- What are the defining equations of $\mathcal{C}_x \subset \mathbb{P}(T_x X)$?
- How varies $\mathcal{C}_x$?

From the adjunction formula, it follows that

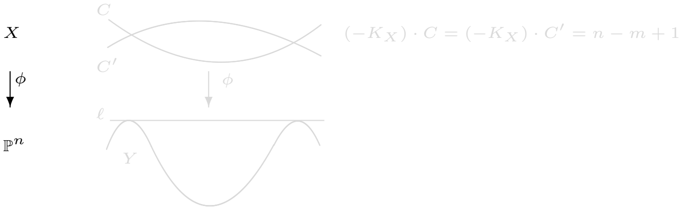
$$K_X = \phi^*(K_{\mathbb{P}^n} + \frac{1}{2}[Y]) = \phi^*(\mathcal{O}_{\mathbb{P}^n}(m-n-1)).$$

Therefore for any curve C in X, the anticanonical degree $(-K_X) \cdot C$ should be a multiple of $n-m+1$.

From the adjunction formula, it follows that

$$K_X = \phi^*(K_{\mathbb{P}^n} + \frac{1}{2}[Y]) = \phi^*(\mathcal{O}_{\mathbb{P}^n}(m-n-1)).$$

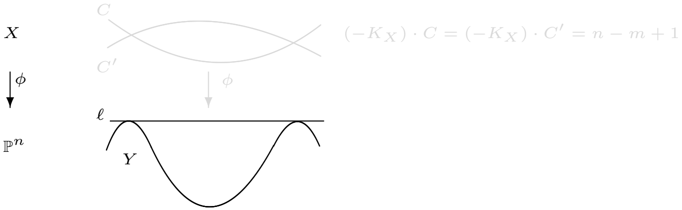
Therefore for any curve C in X, the anticanonical degree $(-K_X) \cdot C$ should be a multiple of $n - m + 1$.



From the adjunction formula, it follows that

$$K_X = \phi^*(K_{\mathbb{P}^n} + \frac{1}{2}[Y]) = \phi^*(\mathcal{O}_{\mathbb{P}^n}(m-n-1)).$$

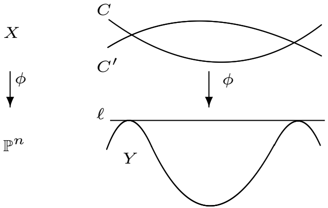
Therefore for any curve C in X, the anticanonical degree $(-K_X) \cdot C$ should be a multiple of $n - m + 1$.



From the adjunction formula, it follows that

$$K_X = \phi^*(K_{\mathbb{P}^n} + \frac{1}{2}[Y]) = \phi^*(\mathcal{O}_{\mathbb{P}^n}(m - n - 1)).$$

Therefore for any curve C in X, the anticanonical degree $(-K_X) \cdot C$ should be a multiple of $n - m + 1$.

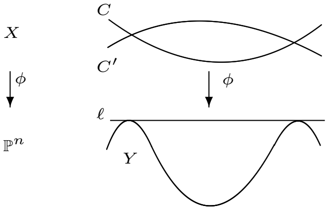


$(-K_X) \cdot C = (-K_X) \cdot C' = n - m + 1$

From the adjunction formula, it follows that

$$K_X = \phi^*(K_{\mathbb{P}^n} + \frac{1}{2}[Y]) = \phi^*(\mathcal{O}_{\mathbb{P}^n}(m-n-1)).$$

Therefore for any curve C in X, the anticanonical degree $(-K_X) \cdot C$ should be a multiple of $n-m+1$.



$(-K_X) \cdot C = (-K_X) \cdot C' = n - m + 1$

For $y \in \mathbb{P}^n - Y$, set

$$\mathcal{E}_y^Y := \{\text{lines } \ell \text{ through } y \text{ such that } \mathrm{mult_z}(\ell \cap \mathrm{Y}) \text{ is even } \forall z \in \ell \cap Y\}.$$

The double covering morphism $\phi : X \to \mathbb{P}^n$ gives an isomorphism

$$d\phi_x : \mathbb{P}(T_x X) \to \mathbb{P}(T_{\phi(x)}\mathbb{P}^n)$$

whose restriction gives an isomorphism

$$\mathcal{C}_x \cong \mathcal{E}^Y_{\phi(x)}.$$

Example

Let $Y \subset \mathbb{P}^n$ be a smooth hypersurface defined by

$$f(t_0, ..., t_n) := t_0^{2m} + t_0^{m-1} b_{m+1} + \cdots + t_0 b_{2m-1} + b_{2m} = 0.$$

Here $b_i = b_i(t_1, ..., t_n)$ is a homogeneous polynomial of deg i. The line passing through $y := (1 : 0 : \cdots : 0) \in \mathbb{P}^n$ and $z := (0 : z_1 : \cdots : z_n)$ belongs to $\mathcal{E}_y^Y$

$$\Leftrightarrow f(1, \lambda z_1, ..., \lambda z_n) = 1 + b_{m+1}(z)\lambda^{m+1} + \cdots + b_{2m}(z)\lambda^{2m}$$

is the square of a polynomial of degree m

$$\Leftrightarrow b_{m+1}(z) = \cdots = b_{2m}(z) = 0$$

Theorem

Let $\phi : X \to \mathbb{P}^n$ *be the double cover branched along a smooth hypersurface* Y *of degree* $2m$, $2 \leq m \leq n-1$.
For general $x \in X$, *the tangent morphism* $\tau_x : \mathcal{K}_x \to \mathcal{C}_x$ *is an isomorphism and*

$$\mathcal{C}_x \subset \mathbb{P}(T_x X) = \mathbb{P}^{n-1}$$

is a smooth complete intersection of degree $(m+1, ..., 2m)$.

Theorem

Let $\phi : X \to \mathbb{P}^n$ be the double cover branched along a smooth hypersurface Y of degree $2m$, $2 \leq m \leq n-1$.
Let $n \geq 4$ and $Y \subset \mathbb{P}^n$ be general. Then the family

$$\{\mathcal{C}_x \subset \mathbb{P}T_x(X) \mid \text{general } x \in X\}$$

has maximal variation.
More precisely, for general $x \in X$, choose a trivialization $\mathbb{P}T(U) \cong \mathbb{P}^{n-1} \times U$ in a neighborhood U of $x \in X$. Define a morphism $\zeta : U \to \mathrm{Hilb}(\mathbb{P}^{n-1})$ by $\zeta(y) := [\mathcal{C}_y]$. Then $\mathrm{rk}(\mathrm{d}\zeta_\mathrm{x}) = \mathrm{n}$ and the intersection of the image of ζ and $GL(n, \mathbb{C})$-orbit of $\zeta(x)$ is isolated at $\zeta(x)$.

Theorem

Let $\phi : X \to \mathbb{P}^n$ be the double cover branched along a smooth hypersurface Y of degree $2m$, $2 \leq m \leq n-1$.
Let $n \geq 4$ and $Y \subset \mathbb{P}^n$ be general. Then the family

$$\{\mathcal{C}_x \subset \mathbb{P}T_x(X) \mid \textit{general } x \in X\}$$

has maximal variation.

More precisely, for general $x \in X$, choose a trivialization $\mathbb{P}T(U) \cong \mathbb{P}^{n-1} \times U$ in a neighborhood U of $x \in X$. Define a morphism $\zeta : U \to \mathrm{Hilb}(\mathbb{P}^{n-1})$ by $\zeta(y) := [\mathcal{C}_y]$. Then $\mathrm{rk}(\mathrm{d}\zeta_\mathrm{x}) = \mathrm{n}$ and the intersection of the image of ζ and $GL(n, \mathbb{C})$-orbit of $\zeta(x)$ is isolated at $\zeta(x)$.

Theorem

Let $\phi : X \to \mathbb{P}^n$ *be the double cover branched along a smooth hypersurface* Y *of degree* $2m$, $2 \leq m \leq n-1$.
Let $n \geq 4$ *and* $Y \subset \mathbb{P}^n$ *be general. Then the family*

$$\{\mathcal{C}_x \subset \mathbb{P}T_x(X) \mid \textit{general } x \in X\}$$

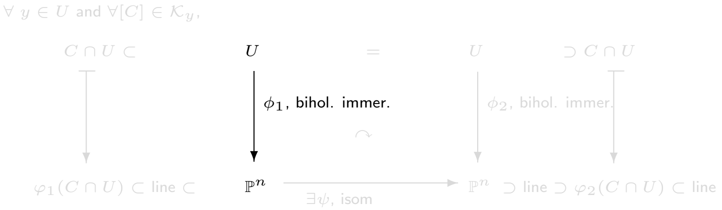
has maximal variation.
More precisely, for general $x \in X$, *choose a trivialization* $\mathbb{P}T(U) \cong \mathbb{P}^{n-1} \times U$ *in a neighborhood* U *of* $x \in X$. *Define a morphism* $\zeta : U \to \mathrm{Hilb}(\mathbb{P}^{n-1})$ *by* $\zeta(y) := [\mathcal{C}_y]$. *Then* $\mathrm{rk}(\mathrm{d}\zeta_{\mathrm{x}}) = \mathrm{n}$ *and the intersection of the image of* ζ *and* $GL(n,\mathbb{C})$*-orbit of* $\zeta(x)$ *is isolated at* $\zeta(x)$.

Theorem

Let X be a Fano manifold such that for general $x \in X$, $\mathcal{C}_x$ is not contained in any quadric hypersurface in $\mathbb{P}T_x(X)$.
Let $U \subset X$ be a connected neighborhood(in classical topology) of a general point $x \in X$ and $\phi_1, \phi_2 : U \to \mathbb{P}^n$ be two biholomorphic immersions such that for any $y \in U$ and any member C of $\mathcal{K}_y$, both $\phi_1(C \cap U)$ and $\phi_2(C \cap U)$ are contained in lines in $\mathbb{P}^n$.
Then there exists a projective transformation $\psi : \mathbb{P}^n \to \mathbb{P}^n$ such that $\phi_2 = \psi \circ \phi_1$.

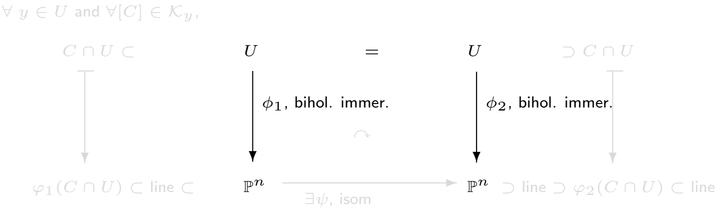
Theorem

Let X be a Fano manifold such that for general $x \in X$, $\mathcal{C}_x$ is not contained in any quadric hypersurface in $\mathbb{P}T_x(X)$.
Let $U \subset X$ be a connected neighborhood(in classical topology) of a general point $x \in X$ and $\phi_1, \phi_2 : U \to \mathbb{P}^n$ be two biholomorphic immersions such that for any $y \in U$ and any member C of $\mathcal{K}_y$, both $\phi_1(C \cap U)$ and $\phi_2(C \cap U)$ are contained in lines in $\mathbb{P}^n$.
Then there exists a projective transformation $\psi : \mathbb{P}^n \to \mathbb{P}^n$ such that $\phi_2 = \psi \circ \phi_1$.



Theorem

Let X be a Fano manifold such that for general $x \in X$, $\mathcal{C}_x$ is not contained in any quadric hypersurface in $\mathbb{P}T_x(X)$.
Let $U \subset X$ be a connected neighborhood(in classical topology) of a general point $x \in X$ and $\phi_1, \phi_2 : U \to \mathbb{P}^n$ be two biholomorphic immersions such that for any $y \in U$ and any member C of $\mathcal{K}_y$, both $\phi_1(C \cap U)$ and $\phi_2(C \cap U)$ are contained in lines in $\mathbb{P}^n$.
Then there exists a projective transformation $\psi : \mathbb{P}^n \to \mathbb{P}^n$ such that $\phi_2 = \psi \circ \phi_1$.



Theorem

Let X be a Fano manifold such that for general $x \in X$, $\mathcal{C}_x$ is not contained in any quadric hypersurface in $\mathbb{P}T_x(X)$.
Let $U \subset X$ be a connected neighborhood(in classical topology) of a general point $x \in X$ and $\phi_1, \phi_2 : U \to \mathbb{P}^n$ be two biholomorphic immersions such that for any $y \in U$ and any member C of $\mathcal{K}_y$, both $\phi_1(C \cap U)$ and $\phi_2(C \cap U)$ are contained in lines in $\mathbb{P}^n$.
Then there exists a projective transformation $\psi : \mathbb{P}^n \to \mathbb{P}^n$ such that $\phi_2 = \psi \circ \phi_1$.

$\forall\, y \in U$ and $\forall [C] \in \mathcal{K}_y$,

$C \cap U \subset \quad U \quad = \quad U \quad \supset C \cap U$

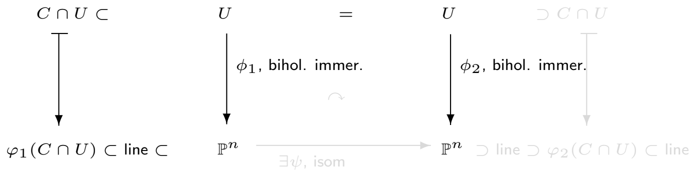
ϕ_1, bihol. immer. $\qquad \phi_2$, bihol. immer.

$\varphi_1(C \cap U) \subset$ line $\subset \quad \mathbb{P}^n \xrightarrow{\exists \psi,\ \text{isom}} \mathbb{P}^n \quad \supset$ line $\supset \varphi_2(C \cap U) \subset$ line

Theorem

Let X be a Fano manifold such that for general $x \in X$, $\mathcal{C}_x$ is not contained in any quadric hypersurface in $\mathbb{P}T_x(X)$.
Let $U \subset X$ be a connected neighborhood(in classical topology) of a general point $x \in X$ and $\phi_1, \phi_2 : U \to \mathbb{P}^n$ be two biholomorphic immersions such that for any $y \in U$ and any member C of $\mathcal{K}_y$, both $\phi_1(C \cap U)$ and $\phi_2(C \cap U)$ are contained in lines in $\mathbb{P}^n$.
Then there exists a projective transformation $\psi : \mathbb{P}^n \to \mathbb{P}^n$ such that $\phi_2 = \psi \circ \phi_1$.

$\forall\, y \in U$ and $\forall [C] \in \mathcal{K}_y$,



Theorem

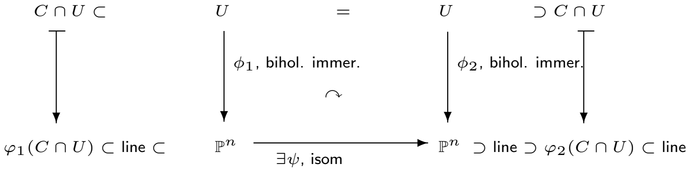
Let X be a Fano manifold such that for general $x \in X$, $\mathcal{C}_x$ is not contained in any quadric hypersurface in $\mathbb{P}T_x(X)$.
Let $U \subset X$ be a connected neighborhood(in classical topology) of a general point $x \in X$ and $\phi_1, \phi_2 : U \to \mathbb{P}^n$ be two biholomorphic immersions such that for any $y \in U$ and any member C of $\mathcal{K}_y$, both $\phi_1(C \cap U)$ and $\phi_2(C \cap U)$ are contained in lines in $\mathbb{P}^n$.
Then there exists a projective transformation $\psi : \mathbb{P}^n \to \mathbb{P}^n$ such that $\phi_2 = \psi \circ \phi_1$.

$\forall\, y \in U$ and $\forall [C] \in \mathcal{K}_y$,

Theorem

Let X be a Fano manifold such that for general $x \in X$, $\mathcal{C}_x$ is not contained in any quadric hypersurface in $\mathbb{P}T_x(X)$.
Let $U \subset X$ be a connected neighborhood(in classical topology) of a general point $x \in X$ and $\phi_1, \phi_2 : U \to \mathbb{P}^n$ be two biholomorphic immersions such that for any $y \in U$ and any member C of $\mathcal{K}_y$, both $\phi_1(C \cap U)$ and $\phi_2(C \cap U)$ are contained in lines in $\mathbb{P}^n$.
Then there exists a projective transformation $\psi : \mathbb{P}^n \to \mathbb{P}^n$ such that $\phi_2 = \psi \circ \phi_1$.

$\forall\, y \in U$ and $\forall [C] \in \mathcal{K}_y$,



Our double cover $\phi : X \to \mathbb{P}^n$ is the first known example of a Fano manifold with Picard number 1 whose VMRT at general point is not contained in any hyperquadric.

We note that

$$\text{for any } [C] \in \mathcal{K}_x,\ \phi(C) \text{ is a line.}$$

Next theorem shows that this property characterizes ϕ in the strong sense.

Our double cover $\phi : X \to \mathbb{P}^n$ is the first known example of a Fano manifold with Picard number 1 whose VMRT at general point is not contained in any hyperquadric.
We note that

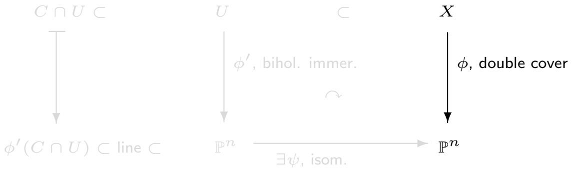
$$\text{for any } [C] \in \mathcal{K}_x,\ \phi(C) \text{ is a line.}$$

Next theorem shows that this property characterizes ϕ in the strong sense.

Theorem

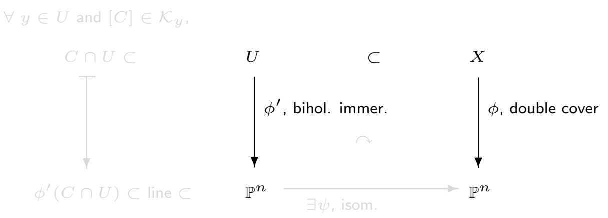
Let $\phi : X \to \mathbb{P}^n$ be the double cover branched along a smooth hypersurface Y of degree $2m$, $2 \le m \le n-1$.
Let $U \subset X$ be a neighborhood (in classical topology) of a general $x \in X$ and let $\phi' : U \to \mathbb{P}^n$ be a biholomorphic immersion such that $\forall\ y \in U$ and $\forall [C] \in \mathcal{K}_y$, $\phi'(C \cap U) \subset$ a line in $\mathbb{P}^n$.
Then there exits an isomorphism $\psi : \mathbb{P}^n \to \mathbb{P}^n$ such that $\phi' = \psi \circ (\phi|_U)$.

$\forall\ y \in U$ and $[C] \in \mathcal{K}_y$,



Theorem

Let $\phi : X \to \mathbb{P}^n$ be the double cover branched along a smooth hypersurface Y of degree $2m$, $2 \leq m \leq n-1$.
Let $U \subset X$ be a neighborhood (in classical topology) of a general $x \in X$ and let $\phi' : U \to \mathbb{P}^n$ be a biholomorphic immersion such that $\forall\ y \in U$ and $\forall [C] \in \mathcal{K}_y$, $\phi'(C \cap U) \subset$ a line in $\mathbb{P}^n$.
Then there exits an isomorphism $\psi : \mathbb{P}^n \to \mathbb{P}^n$ such that $\phi' = \psi \circ (\phi|_U)$.



Theorem

Let $\phi : X \to \mathbb{P}^n$ be the double cover branched along a smooth hypersurface Y of degree $2m$, $2 \leq m \leq n-1$.
Let $U \subset X$ be a neighborhood (in classical topology) of a general $x \in X$ and let $\phi' : U \to \mathbb{P}^n$ be a biholomorphic immersion such that $\forall\ y \in U$ and $\forall [C] \in \mathcal{K}_y$, $\phi'(C \cap U) \subset$ a line in $\mathbb{P}^n$.
Then there exits an isomorphism $\psi : \mathbb{P}^n \to \mathbb{P}^n$ such that $\phi' = \psi \circ (\phi|_U)$.

$\forall\ y \in U$ and $[C] \in \mathcal{K}_y$,

$$\begin{array}{ccccc} C \cap U \subset & U & \subset & X \\ \downarrow & \downarrow \phi', \text{ bihol. immer.} & \circlearrowright & \downarrow \phi, \text{ double cover} \\ \phi'(C \cap U) \subset \text{line} \subset & \mathbb{P}^n & \xrightarrow[\exists \psi, \text{ isom.}]{} & \mathbb{P}^n \end{array}$$

Theorem

Let $\phi : X \to \mathbb{P}^n$ be the double cover branched along a smooth hypersurface Y of degree $2m$, $2 \leq m \leq n-1$.
Let $U \subset X$ be a neighborhood (in classical topology) of a general $x \in X$ and let $\phi' : U \to \mathbb{P}^n$ be a biholomorphic immersion such that $\forall\ y \in U$ and $\forall [C] \in \mathcal{K}_y$, $\phi'(C \cap U) \subset$ a line in $\mathbb{P}^n$.
Then there exits an isomorphism $\psi : \mathbb{P}^n \to \mathbb{P}^n$ such that $\phi' = \psi \circ (\phi|_U)$.

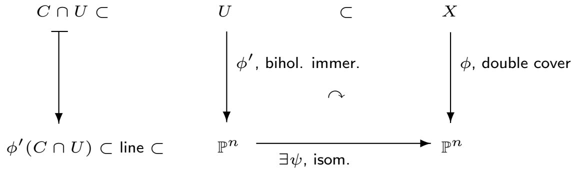
$\forall\ y \in U$ and $[C] \in \mathcal{K}_y$,

$$
\begin{array}{ccccc}
C \cap U \subset & U & \subset & X \\
\big\downarrow & \big\downarrow \phi', \text{ bihol. immer.} & \circlearrowright & \big\downarrow \phi, \text{ double cover} \\
\phi'(C \cap U) \subset \text{line} \subset & \mathbb{P}^n & \xrightarrow[\exists \psi, \text{ isom.}]{} & \mathbb{P}^n
\end{array}
$$

Theorem

Let $\phi : X \to \mathbb{P}^n$ be the double cover branched along a smooth hypersurface Y of degree $2m$, $2 \le m \le n-1$.
Let $U \subset X$ be a neighborhood (in classical topology) of a general $x \in X$ and let $\phi' : U \to \mathbb{P}^n$ be a biholomorphic immersion such that $\forall\ y \in U$ and $\forall [C] \in \mathcal{K}_y$, $\phi'(C \cap U) \subset$ a line in $\mathbb{P}^n$.
Then there exits an isomorphism $\psi : \mathbb{P}^n \to \mathbb{P}^n$ such that $\phi' = \psi \circ (\phi|_U)$.

$\forall\ y \in U$ and $[C] \in \mathcal{K}_y$,



Theorem

Let $Y_1, Y_2 \subset \mathbb{P}^n, n \geq 3$, be two smooth hypersurfaces of degree $2(n-1)$. Let $\phi_1 : X_1 \to \mathbb{P}^n$, $\phi_2 : X_2 \to \mathbb{P}^n$ be two double covers of $\mathbb{P}^n$ branched along Y_1 and Y_2, respectively. Suppose there exists a finite morphism $f : X_1 \to X_2$. Then f is an isomorphism.

$\Rightarrow$ f is an isomorphism

Theorem

Let $Y_1, Y_2 \subset \mathbb{P}^n, n \geq 3$, be two smooth hypersurfaces of degree $2(n-1)$. Let $\phi_1 : X_1 \to \mathbb{P}^n$, $\phi_2 : X_2 \to \mathbb{P}^n$ be two double covers of $\mathbb{P}^n$ branched along Y_1 and Y_2, respectively. Suppose there exists a finite morphism $f : X_1 \to X_2$. Then f is an isomorphism.

Theorem

Let $Y_1, Y_2 \subset \mathbb{P}^n, n \geq 3,$ be two smooth hypersurfaces of degree $2(n-1)$. Let $\phi_1 : X_1 \to \mathbb{P}^n$, $\phi_2 : X_2 \to \mathbb{P}^n$ be two double covers of $\mathbb{P}^n$ branched along Y_1 and Y_2, respectively. Suppose there exists a finite morphism $f : X_1 \to X_2$. Then f is an isomorphism.

$$\begin{array}{ccc} X_1 & \xrightarrow{f\text{, finite morph.}} & X_2 \\ \downarrow {\scriptstyle \phi_1\text{, double cover}} & & \downarrow {\scriptstyle \phi_2\text{, double cover}} \\ \mathbb{P}^n & = & \mathbb{P}^n \end{array}$$

$\Rightarrow$ f is an isomorphism

Theorem

Let $Y_1, Y_2 \subset \mathbb{P}^n, n \geq 3$, be two smooth hypersurfaces of degree $2(n-1)$. Let $\phi_1 : X_1 \to \mathbb{P}^n$, $\phi_2 : X_2 \to \mathbb{P}^n$ be two double covers of $\mathbb{P}^n$ branched along Y_1 and Y_2, respectively. Suppose there exists a finite morphism $f : X_1 \to X_2$. Then f is an isomorphism.

$$
\begin{array}{ccc}
X_1 & \xrightarrow{f\text{, finite morph.}} & X_2 \\
\Big\downarrow {\scriptstyle \phi_1\text{, double cover}} & & \Big\downarrow {\scriptstyle \phi_2\text{, double cover}} \\
\mathbb{P}^n & = & \mathbb{P}^n
\end{array}
$$

$\Rightarrow$ f is an isomorphism

Theorem

Let $Y_1, Y_2 \subset \mathbb{P}^n, n \geq 3$, be two smooth hypersurfaces of degree $2(n-1)$. Let $\phi_1 : X_1 \to \mathbb{P}^n$, $\phi_2 : X_2 \to \mathbb{P}^n$ be two double covers of $\mathbb{P}^n$ branched along Y_1 and Y_2, respectively. Suppose there exists a finite morphism $f : X_1 \to X_2$. Then f is an isomorphism.

$$\begin{array}{ccc} X_1 & \xrightarrow{f\text{, finite morph.}} & X_2 \\ \downarrow {\scriptstyle \phi_1\text{, double cover}} & & \downarrow {\scriptstyle \phi_2\text{, double cover}} \\ \mathbb{P}^n & = & \mathbb{P}^n \end{array} \quad \Rightarrow \text{ f is an isomorphism}$$

Problem (Liouville-type extension problem)

Let X be a Fano manifold of Picard number 1. Let U_1 and U_2 be two connected open subsets (in classical topology) in X. Suppose that we are given a biholomorphic map $\gamma : U_1 \to U_2$ such that for any minimal rational curve $C \subset X$, there exists another minimal rational curve C' with $\gamma(U_1 \cap C) = U_2 \cap C'$. Then does there exist $\Gamma \in \mathrm{Aut}(X)$ with $\Gamma|_{U_1} = \gamma$?

$$
\begin{array}{ccc}
U_1 & \xrightarrow{\gamma,\ \text{biholom.}} & U_2 \\
\cap & \circlearrowright & \cap \\
X & \xrightarrow[\exists\, \Gamma,\ \text{biholom}]{} & X
\end{array}
$$

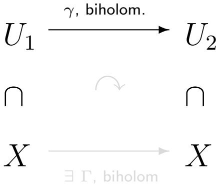
Problem (Liouville-type extension problem)

Let X be a Fano manifold of Picard number 1. Let U_1 and U_2 be two connected open subsets (in classical topology) in X. Suppose that we are given a biholomorphic map $\gamma : U_1 \to U_2$ such that for any minimal rational curve $C \subset X$, there exists another minimal rational curve C' with $\gamma(U_1 \cap C) = U_2 \cap C'$. Then does there exist $\Gamma \in \mathrm{Aut}(X)$ with $\Gamma|_{U_1} = \gamma$?

$$\begin{array}{ccc} U_1 & \xrightarrow{\gamma,\ \text{biholom.}} & U_2 \\ \cap & \circlearrowright & \cap \\ X & \xrightarrow[\exists\ \Gamma,\ \text{biholom}]{} & X \end{array}$$

Problem (Liouville-type extension problem)

Let X be a Fano manifold of Picard number 1. Let U_1 and U_2 be two connected open subsets (in classical topology) in X. Suppose that we are given a biholomorphic map $\gamma : U_1 \to U_2$ such that for any minimal rational curve $C \subset X$, there exists another minimal rational curve C' with $\gamma(U_1 \cap C) = U_2 \cap C'$. Then does there exist $\Gamma \in \mathrm{Aut}(X)$ with $\Gamma|_{U_1} = \gamma$?



Problem (Liouville-type extension problem)

Let X be a Fano manifold of Picard number 1. Let U_1 and U_2 be two connected open subsets (in classical topology) in X. Suppose that we are given a biholomorphic map $\gamma : U_1 \to U_2$ such that for any minimal rational curve $C \subset X$, there exists another minimal rational curve C' with $\gamma(U_1 \cap C) = U_2 \cap C'$. Then does there exist $\Gamma \in \mathrm{Aut}(X)$ with $\Gamma|_{U_1} = \gamma$?

$$\begin{array}{ccc} U_1 & \xrightarrow{\gamma,\ \text{biholom.}} & U_2 \\ \cap & \circlearrowright & \cap \\ X & \xrightarrow[\exists\, \Gamma,\ \text{biholom}]{} & X \end{array}$$

Problem (Liouville-type extension problem)

Let X be a Fano manifold of Picard number 1. Let U_1 and U_2 be two connected open subsets (in classical topology) in X. Suppose that we are given a biholomorphic map $\gamma : U_1 \to U_2$ such that for any minimal rational curve $C \subset X$, there exists another minimal rational curve C' with $\gamma(U_1 \cap C) = U_2 \cap C'$. Then does there exist $\Gamma \in \mathrm{Aut}(X)$ with $\Gamma|_{U_1} = \gamma$?

$$\begin{array}{ccc} U_1 & \xrightarrow{\gamma,\ \text{biholom.}} & U_2 \\ \cap & \curvearrowright & \cap \\ X & \xrightarrow[\exists\ \Gamma,\ \text{biholom}]{} & X \end{array}$$

Theorem

Let $Y_1, Y_2 \subset \mathbb{P}^n, n \geq 3$, be two smooth hypersurfaces of degree $2m, 2 \leq m \leq n-1$. Let $\phi_1 : X_1 \to \mathbb{P}^n$, $\phi_2 : X_2 \to \mathbb{P}^n$ be double covers of $\mathbb{P}^n$ branched along Y_1 and Y_2, respectively. Let $U_1 \subset X_1$, $U_2 \subset X_2$ be connected open subsets. Suppose that we are given a biholomorphic map $\gamma : U_1 \to U_2$ be a biholomorphic map such that for any minimal rational curve $C_1 \subset X_1$, there exists a minimal rational curve $C_2 \subset X_2$ with $\gamma(U_1 \cap C_1) = U_2 \cap C_2$. Then $\exists$ a biregular morphism $\Gamma : X_1 \to X_2$ with $\Gamma|_{U_1} = \gamma$.

$\forall\, C_1$ minimal rational curve $\qquad \exists\, C_2$ minimal rational curve

$$
\begin{array}{ccc}
C_1 \cap U_1 & \mapsto & C_2 \cap U_2 \\
\cap & & \cap \\
U_1 & \xrightarrow{\gamma,\ \text{biholom.}} & U_2 \\
\cap & \curvearrowright & \cap \\
X_1 & \xrightarrow[\exists\,\Gamma,\ \text{isom}]{} & X_2 \\
\downarrow \phi_1,\ \text{double cover} & & \downarrow \phi_2,\ \text{double cover} \\
\mathbb{P}^n & & \mathbb{P}^n
\end{array}
$$

Thank you!

$\forall\ C_1$ minimal rational curve $\qquad \exists\ C_2$ minimal rational curve

$$
\begin{array}{ccc}
C_1 \cap U_1 & \mapsto & C_2 \cap U_2 \\
\cap & & \cap \\
U_1 & \xrightarrow{\gamma,\ \text{biholom.}} & U_2 \\
\cap & \curvearrowright & \cap \\
X_1 & \xrightarrow[\exists\ \Gamma,\ \text{isom}]{} & X_2 \\
\downarrow \phi_1,\ \text{double cover} & & \downarrow \phi_2,\ \text{double cover} \\
\mathbb{P}^n & & \mathbb{P}^n
\end{array}
$$

Thank you!

$\forall\ C_1$ minimal rational curve $\qquad \exists\ C_2$ minimal rational curve

$$C_1 \cap U_1 \mapsto C_2 \cap U_2$$

$$\begin{array}{ccc}
\cap & & \cap \\
U_1 & \xrightarrow{\gamma,\ \text{biholom.}} & U_2 \\
\cap & \circlearrowright & \cap \\
X_1 & \xrightarrow[\exists\ \Gamma,\ \text{isom}]{} & X_2 \\
\big\downarrow {\scriptstyle \phi_1,\ \text{double cover}} & & \big\downarrow {\scriptstyle \phi_2,\ \text{double cover}} \\
\mathbb{P}^n & & \mathbb{P}^n
\end{array}$$

Thank you!

$\forall\, C_1$ minimal rational curve $\qquad \exists\, C_2$ minimal rational curve

$$\begin{array}{ccc}
C_1 \cap U_1 & \mapsto & C_2 \cap U_2 \\
\cap & & \cap \\
U_1 & \xrightarrow{\gamma,\ \text{biholom.}} & U_2 \\
\cap & \circlearrowright & \cap \\
X_1 & \xrightarrow[\exists\, \Gamma,\ \text{isom}]{} & X_2 \\
\downarrow{\scriptstyle \phi_1,\ \text{double cover}} & & \downarrow{\scriptstyle \phi_2,\ \text{double cover}} \\
\mathbb{P}^n & & \mathbb{P}^n
\end{array}$$

Thank you!

$\forall\, C_1$ minimal rational curve $\qquad \exists\, C_2$ minimal rational curve

$$
\begin{array}{ccc}
C_1 \cap U_1 & \mapsto & C_2 \cap U_2 \\
\cap & & \cap \\
U_1 & \xrightarrow{\gamma,\ \text{biholom.}} & U_2 \\
\cap & \circlearrowright & \cap \\
X_1 & \xrightarrow[\exists\, \Gamma,\ \text{isom}]{} & X_2 \\
\downarrow {\scriptstyle \phi_1,\ \text{double cover}} & & \downarrow {\scriptstyle \phi_2,\ \text{double cover}} \\
\mathbb{P}^n & & \mathbb{P}^n
\end{array}
$$

Thank you!

$\forall\ C_1$ minimal rational curve $\qquad \exists\ C_2$ minimal rational curve

$$
\begin{array}{ccc}
C_1 \cap U_1 & \mapsto & C_2 \cap U_2 \\
\cap & & \cap \\
U_1 & \xrightarrow{\gamma,\ \text{biholom.}} & U_2 \\
\cap & \circlearrowright & \cap \\
X_1 & \xrightarrow[\exists\ \Gamma,\ \text{isom}]{} & X_2 \\
\downarrow {\scriptstyle \phi_1,\ \text{double cover}} & & \downarrow {\scriptstyle \phi_2,\ \text{double cover}} \\
\mathbb{P}^n & & \mathbb{P}^n
\end{array}
$$

Thank you!

$\forall\ C_1$ minimal rational curve $\qquad \exists\ C_2$ minimal rational curve

$$
\begin{array}{ccc}
C_1 \cap U_1 & \mapsto & C_2 \cap U_2 \\
\cap & & \cap \\
U_1 & \xrightarrow{\gamma,\ \text{biholom.}} & U_2 \\
\cap & \curvearrowright & \cap \\
X_1 & \xrightarrow[\exists\ \Gamma,\ \text{isom}]{} & X_2 \\
\downarrow{\scriptstyle \phi_1,\ \text{double cover}} & & \downarrow{\scriptstyle \phi_2,\ \text{double cover}} \\
\mathbb{P}^n & & \mathbb{P}^n
\end{array}
$$

Thank you!